PRINCIPAL RESPONSE ANALYSIS IN STRUCTURAL DYNAMICS

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Abstract: This presentation describes two new Mode Indicator Functions (MIFs) based on the Principal Response Analysis (PRA) of Frequency Response Function (FRF) test data. Instead of the stepwise analysis of rectangular FRF matrices, one frequency at a time, as in some MIFs in practical use, herein a simultaneous analysis is carried out of all FRF information in a single matrix, encompassing all frequency points. The Singular Value Factorization (SVF) of such Compound FRF (CFRF) multi-frequency matrices helps separating the frequency dependence from the spatial dependence of FRF data. The analysis of SVF-related quantities calculated for CFRF matrices can be used to determine the number of modes present in a given frequency range, to identify (quasi-) repeated natural frequencies and to pre-process the FRF data to make them more amenable to the modal analysis.
PRA provides a reduced set of uncorrelated Principal Response Functions (PRFs), computed as linear combinations of the measured FRFs. The frequency dependence of the PRFs is used to define two new modal indicator functions able to estimate the number of active modes of vibration even from an incomplete data set. Analysing a selected peak of each PRF allows determination of modal parameters by single degree of freedom identification techniques.

INTRODUCTION

The Principal Response Analysis (PRA) is used to express the dynamic response of a structure in terms of a few linear combinations of the original Frequency Response Functions (FRF), referred to as Principal Response Functions (PRFs). These are helpful in separating data from noise, eliminating redundant linearly-dependent information, and estimating the rank and condition of test data. Analysis of selected peaks of PRFs allows identification of natural frequencies and modal damping values.

Consider a set of test data in the form of N complex valued FRFs sampled at N_f frequencies, arranged columnwise in a Compound FRF (CFRF) matrix, $\boldsymbol{A} \in \mathbf{C}^{N_f x N}$. Each column, $\boldsymbol{a}_i$, contains the frequency-dependent elements of an individual FRF, measured at a given output/input co-ordinate combination. Each row contains N complex FRF values measured at the same frequency.

Although N FRFs (components) are apparently required to reproduce the total system dynamic behaviour, often much of this behaviour can be accounted for by a small number, N_r, of principal responses. If so, there is (almost) as much information in the N_r response components as there is in the original N FRFs. The N_r principal components can replace the initial (measured) N FRFs. The original data set, consisting of N_f measurements on N variables, is reduced to one consisting of N_f discrete values of N_r principal components.

Due to noise and non-linear effects, the CFRF matrix is apparently of full rank. An effective rank can be estimated looking for gaps in the magnitude of its singular values. Using the plot of the ratio of successive singular values, the rank of the FRF matrix is set to the index of the value for which the ratio is a minimum.

Moreover, analysis of all FRF test data, arranged in a single multi-frequency CFRF matrix, can smear the frequency shifts and other inaccuracies due to the non-simultaneous measurement of the FRFs.

PRINCIPAL RESPONSE ANALYSIS OF THE CFRF MATRIX

Let the Singular Value Factorization (SVF) of the CFRF matrix be written as

$$\boldsymbol{A} = \boldsymbol{U}\,\boldsymbol{\Sigma}\,\boldsymbol{V}^{\mathrm{H}} = \sum_{i=1}^{N} \sigma_i\, \boldsymbol{u}_i\, \boldsymbol{v}_i^{\mathrm{H}} = \sum_{i=1}^{N} \boldsymbol{A}_i \qquad (1)$$

where $\boldsymbol{U} \in \mathbf{C}^{N_f x N}$ and $\boldsymbol{V} \in \mathbf{C}^{NxN}$ are the matrices of the left and right singular vectors, respectively, and the superscript H denotes the conjugate transpose (Hermitian). The singular values σ_i are arranged in non-increasing order in the real diagonal matrix $\boldsymbol{\Sigma}$.

Because the left and right singular vectors have unit length, the amplitude information is contained in the singular values. The SVF decomposes the CFRF matrix into a sum of rank-one matrices $\boldsymbol{A}_i = \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^H$ of the same size as $\boldsymbol{A}$. Each singular value is equal to the Frobenius norm of the associated $\boldsymbol{A}_i$ matrix

$$\sigma_i = \left\| \boldsymbol{A}_i \right\|_F \qquad (2)$$

and can be considered as a measure of its energy content.

The left singular vectors (LSV), referred to as principal components,

$$\boldsymbol{u}_i = \frac{1}{\sigma_i} \sum_{j=1}^{N} v_{ji} \boldsymbol{a}_j \qquad (3)$$

contain the frequency distribution of the energy. The right singular vectors (RSV), $\boldsymbol{v}_i$, describe the spatial distribution of the energy contained in the FRF set.

The Principal Response Functions, $\boldsymbol{p}_i$, defined as the LSVs scaled by the respective singular values [1], are linear combinations of the original FRFs, $\boldsymbol{a}_i$:

$$\boldsymbol{p}_i = \sigma_i \boldsymbol{u}_i = \boldsymbol{A}\,\boldsymbol{v}_i = \sum_{j=1}^{N} v_{ji} \boldsymbol{a}_j \ . \qquad (4)$$

Transforming the original FRFs to PRFs amounts to a rotation of coordinate axes to a new coordinate system that has inherent energy properties. The PRFs give a new set of linearly combined measurements. PRFs are orthogonal vectors, each one representing the frequency distribution of an amount of energy equal to the square of the related singular value.

The matrix having the PRFs as columns is

$$\boldsymbol{P} = \boldsymbol{A}\boldsymbol{V} = \boldsymbol{U}\,\boldsymbol{\Sigma}\,. \tag{5}$$

If $\boldsymbol{A}$ is rank-deficient, then a reduced number of columns is retained in the matrix $\boldsymbol{P}$.

In equation 4 the multiplying factors v_{ji} are the complex valued elements of the right singular vectors. PRFs are also the eigenvectors of the matrix $\boldsymbol{A}\boldsymbol{A}^H$ scaled by the square roots of the respective eigenvalues. Like the LSV they are mutually (pairwise) orthogonal vectors, so they are linearly independent.

Algebraically, Principal Response Functions are particular linear combinations of the initial (measured) N FRFs.

Geometrically, these linear combinations represent the selection of a new coordinate system, obtained by rotating the original system with $\boldsymbol{a}_i$ as the coordinate axes. The unit vectors along these directions are the LSVs which are orthonormal.

Physically, the new axes represent the directions with maximum power (energy) and provide a simpler and more parsimonious description of the FRF data.

Indeed, if $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{A}^H$ is the cross-power matrix (power spectrum matrix), its spectral decomposition is

$$\boldsymbol{A}\,\boldsymbol{A}^H\,\boldsymbol{U} = \boldsymbol{U}\,\boldsymbol{\Sigma}^2 \tag{6}$$

One can think of

$$\int_{t_1}^{t_2} \left\| x(t) \right\|^2 \, \mathrm{d}t$$

as the total energy in the response set $\{\boldsymbol{x}(t)\}$ over the interval $[t_1 \;\; t_2]$.

The Frobenius norm of the CFRF matrix is

$$\left\|\boldsymbol{A}\right\|_F = \left(\sum_{i=1}^{N_f} \sum_{j=1}^{N_f} \left| c_{ij} \right|^2 \right)^{1/2} . \tag{7}$$

The trace of $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{A}^H$ is related to the total power

$$\left\|\boldsymbol{A}\right\|_F^2 = \mathrm{tr}\left(\boldsymbol{A}\boldsymbol{A}^{\mathrm{H}}\right) = \mathrm{tr}\left(\boldsymbol{U}\,\boldsymbol{\Sigma}^2\,\boldsymbol{U}^{\mathrm{H}}\right) = \mathrm{tr}\left(\boldsymbol{U}^{\mathrm{H}}\boldsymbol{U}\,\boldsymbol{\Sigma}^2\right) = \mathrm{tr}\left(\boldsymbol{\Sigma}^2\right) = \sum_{i=1}^{N_f} \sigma_i^2 . \tag{8}$$

Because, by definition, $\sigma_1 \geq \sigma_2 \geq \ldots\ldots \geq \sigma_r$, the SVF decomposes the CFRF matrix $\boldsymbol{A}$ into a sum of principal component matrices containing decreasing levels of energy.

So, one can say that, for example, the first LSV is the normalised linear combination of original FRFs (columns of the CFRF matrix) with maximum energy (power) content.

In terms of the matrix of PRFs

$$\boldsymbol{A}\,\boldsymbol{A}^{\mathrm{H}} = \boldsymbol{U}\,\boldsymbol{\Sigma}^{2}\boldsymbol{U}^{\mathrm{H}} = \boldsymbol{U}\,\boldsymbol{\Sigma}\,\boldsymbol{\Sigma}\,\boldsymbol{U}^{\mathrm{H}} = \boldsymbol{P}\,\boldsymbol{P}^{\mathrm{H}} \tag{9}$$

so that

$$\mathrm{tr}\left(\boldsymbol{A}\boldsymbol{A}^{\mathrm{H}}\right) = \mathrm{tr}\left(\boldsymbol{P}\boldsymbol{P}^{\mathrm{H}}\right) = \sum_{i=1}^{N_r} \sigma_i^2 \tag{10}$$

In many cases, the number of measured FRFs is too large, the FRF data set contains redundant information, FRFs being not linearly independent. A way of reducing the number of FRFs to be analysed is to discard the linear combinations of FRFs which have small energy and study only those with large energies.

Otherwise stated, the cross-product matrix of the uncorrelated PRFs should be diagonal, representing autopower. Indeed,

$$\boldsymbol{P}^{H}\boldsymbol{P} = \left(\boldsymbol{A}\boldsymbol{V}\right)^{H}\boldsymbol{A}\boldsymbol{V} = \boldsymbol{V}^{H}\left(\boldsymbol{A}^{H}\boldsymbol{A}\right)\boldsymbol{V} = \boldsymbol{\Sigma}^{2}, \tag{11}$$

so that the cross-power off-diagonal elements are zero. The columns of **P** form a set of orthogonal response functions, each one representing an amount of energy equal to the square of the related singular value. The first PRF, corresponding to the largest singular value, is the uncorrelated response function with the largest autopower. The second PRF has the second largest autopower, and so on.

PRFs have peaks at the natural frequencies, as have the FRFs. The modes whose shape is similar to the weighting RSV are enhanced, while the others are attenuated. For an adequate selection of input/output coordinate combinations, each PRF is dominated by a single mode of vibration. Single degree of freedom identification techniques can be used to determine the corresponding modal parameters [2]. For a non-optimal location of sensors and excitation coordinates, resulting in an insufficient spatial independence of the modal vectors, and for limited spatial resolution, a PRF can have multiple peaks, especially when this is backed by insufficient frequency resolution.

The plot of left singular vectors versus frequency helps locating the natural frequencies, but is rather confusing for noise polluted data. A similar plot of PRFs is more useful. PRFs with low energy level are LSVs multiplied by small singular values, so the respective curves are shifted down. A gap in the singular values produces a marked separation of PRFs containing useful information from those with negligible energy content and polluted by noise. If the first N_r PRFs are separated from the others in the upper part of the PRF plot, then N_r can be chosen as the effective rank of the CFRF matrix. However, the vertical shifting of the PRF curves, due to the multiplication of LSVs by the singular values, can obscure the highest peaks, if not located by distinctive marks.

MODE INDICATOR FUNCTIONS

The Componentwise Mode Indicator Function (CoMIF)

For each component $\boldsymbol{A}_i$ of the CFRF matrix, the diagonal elements of the orthogonal projector onto the null space of $\boldsymbol{A}_i^H$ exhibit minima at the natural frequencies.

The Componentwise Mode Indicator Function (CoMIF) is defined [3] by vectors of the form:

$$\boldsymbol{CoMIF}_i = \mathrm{diag}\left(\boldsymbol{I}_{N_f} - \boldsymbol{A}_i \boldsymbol{A}_i^{+}\right) = \mathrm{diag}\left(\boldsymbol{I}_{N_f} - \boldsymbol{u}_i \boldsymbol{u}_i^{\mathrm{H}}\right), \tag{12}$$

where **+** denotes the pseudoinverse, and $\boldsymbol{I}_{N_f}$ is the identity matrix of order N_f.

In the CoMIF plot, for each principal component, the diagonal elements of the projector onto the plane perpendicular to the axis, having the respective LSV as the unit vector, are displayed against frequency. The CoMIF is a multi-curve mode indicator that can locate both repeated modal frequencies and highly damped close modes. The number of curves is equal to the estimated effective rank of the CFRF matrix, i.e. to the truncated number of its principal components. Each curve has a local minimum at a natural frequency, with the deepest trough at the natural frequency of the corresponding dominant mode. The CoMIF curves are not related to the number of excitation points, as in the case of other MIFs, but their accuracy in locating natural frequencies is strongly dependent on the spatial distribution of the excitation forces and response measurement transducers.

Visual inspection of CoMIF curves reveals the number of modes active in a given frequency band and the dominant mode in each CoMIF curve, which is the strongest enhanced mode in the respective PRF. The componentwise analysis allows a better estimation of the rank of the CFRF matrix and a better understanding of the contribution of each mode to the dynamics of the system.

The PRF Mode Indicator Function (PRF MIF)

For noisy data and for structures with high modal density, an overlay of the CoMIF curves becomes hard to interpret, so that a single-curve mode indicator function has been developed.

The PRF Mode Indicator Function (PRFMIF), defined as the AMIF in [4], is:

$$\boldsymbol{PRFMIF} = \mathrm{diag}\,(\boldsymbol{A}\boldsymbol{A}^{+}). \tag{13}$$

If $\boldsymbol{A}$ is rank-deficient and its effective rank is N_r, then a rank-limited matrix $\tilde{\boldsymbol{A}}$, referred to as the Aggregate FRF matrix, can be constructed retaining only the N_r largest principal components. The PRFMIF becomes:

$$\boldsymbol{PRFMIF} = \mathrm{diag}\,(\tilde{\boldsymbol{A}}\tilde{\boldsymbol{A}}^{+}) = \mathrm{diag}\left(\sum_{i=1}^{N_r} \boldsymbol{u}_i \boldsymbol{u}_i^{\mathrm{H}}\right). \tag{14}$$

Different PRFMIF curves can be plotted for different values of N_r, as in [5]. From equation 14 it is seen that the PRFMIF is an aggregate of vectors of the form

$$\boldsymbol{PRFMIF}_i = \mathrm{diag}\,(\boldsymbol{A}_i \boldsymbol{A}_i^{+}) = \mathrm{diag}\left(\boldsymbol{u}_i \boldsymbol{u}_i^{\mathrm{H}}\right) \tag{15}$$

and the sum extends over a number of principal components equal to the estimated rank of the CFRF matrix.

Applied to the CFRF matrix, the PRFMIF reveals local modes not shown by other mode indicators. Any difference between the number of peaks in the PRFMIF and the rank of the CFRF matrix indicates either some contribution from out-of-band modes, or the existence of double modes.

The PRFMIF and CoMIF are currently implemented in the MODENT Suite of software for modal analysis by ICATS [6].

Other Mode Indicator Functions

The CoMIF and the PRFMIF are compared with two currently used mode indicator functions, the MIF and the ImMIF.

At each frequency the MIF is defined [7] as

$$MIF_i = 1 - \frac{\sum_{j=1}^{N} \left| Re(a_{ij}) \right| \left| a_{ij} \right|}{\sum_{j=1}^{N} \left| a_{ij} \right|^2}, \tag{16}$$

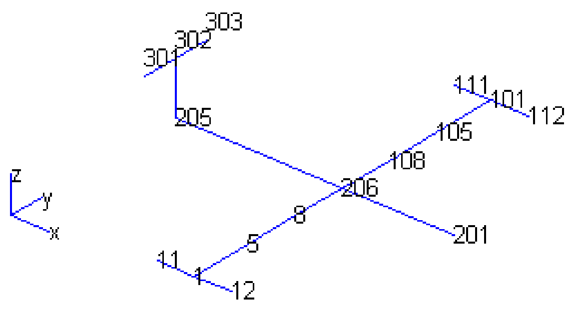
while the somewhat complementary ImMIF is defined [8] as

$$ImMIF_i = 1 - \frac{\sum_{j=1}^{N} \left| Im(a_{ij}) \right| \left| a_{ij} \right|}{\sum_{j=1}^{N} \left| a_{ij} \right|^2}. \tag{17}$$

The last two locate the frequencies where the forced response is closest to the monophase condition. The first two also locate modes with a higher degree of complexity, being based on the information density in the FRF data set.

GARTEUR SM-AG 19 TESTBED RESULTS

The PRA technique and the new MIFs were tested on the FRF data set referred to as UNMOD – for the unmodified GARTEUR structure as reassembled in the Centre of Vibration Engineering, at the Imperial College, London.



Excitation DOFs			
101z	105x	105z	108z
111z	112x	112z	11z
12x	12z	1z	201x
201y	201z	205y	206z
301x	301z	302y	303x
303z	5x	5z	8z
Response DOF: 12z			

Fig. 1: The measurement locations

The experimental data-base consists of 24 complex valued FRFs, measured at 24 arbitrarily selected locations (Fig.1) using single point hammer excitation and response measured at the right wing tip. The data set spans a frequency range from about 0 to 100 Hz, with 0.125 Hz frequency resolution.

Using the FRFs from the UNMOD data set, a CFRF matrix of size 801x24 has been constructed. The first eleven PRFs are shown in Fig.2. An overlay of the CoMIF curves is presented in Fig.3. As in Fig.2, the number near each local minimum indicates the index of the CoMIF curve (hence of the PRF) where the respective mode is dominant.

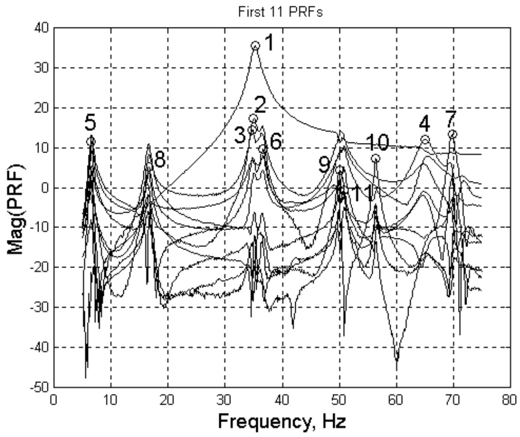


Fig. 2: First 11 PRFs

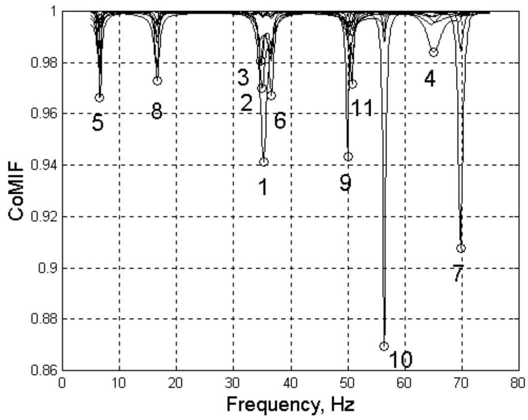


Fig.3: Overlaid CoMIFs

The CoMIF plot (Fig.4) clearly reveals 10 modes of vibration between 5 and 75 Hz. Subplots correspond to separate CoMIFs with the index shown on the left. Each detected mode is marked by a local minimum at the associated frequency. The deepest minimum indicates the dominant mode.

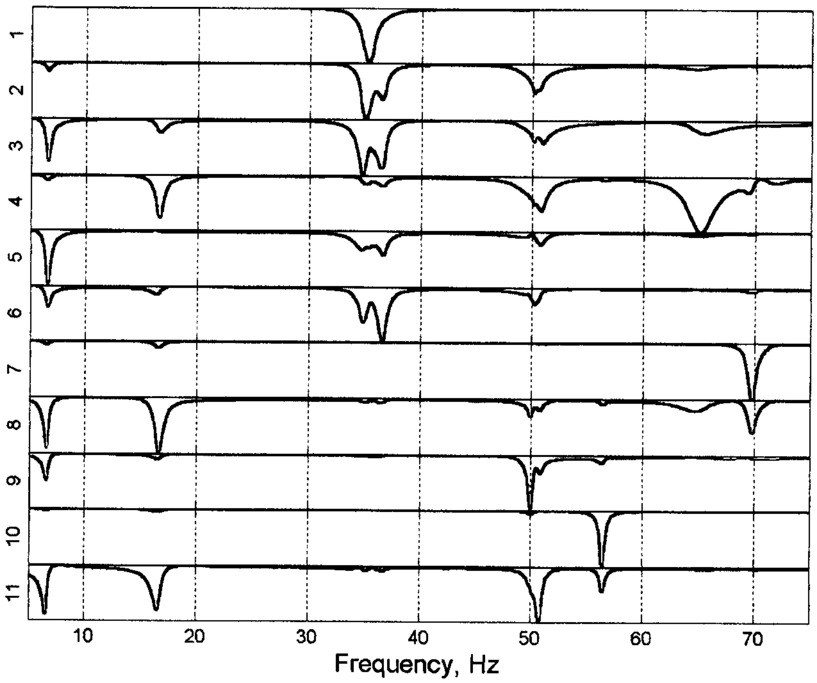


Fig. 4: CoMIF plot

Table 1 lists the natural frequencies and (equivalent viscous) damping ratios determined by SDOF circle fit. The index of the PRF used for modal parameter identification is given in column 2.

Table 1: Modal parameters for unmodified GARTEUR structure

Mode	PRF	Natural frequency, Hz	Damping ratio, %	Description
1	5	6.55	4.13	2N wing bending
2	8	16.60	2.60	Fuselage rotation
3	3	34.91	1.03	Antisym. wing torsion
4	1	35.30	1.90	Symmetric wing torsion
5	6	36.54	1.20	3N wing bending
6	9	49.99	0.49	4N wing bending
7	11	50.76	0.75	Inpl. wing vs. fuselage
8	10	56.43	0.45	Sym. in-plane wing bending
9	4	65.04	2.21	5N wing bending
10	7	69.72	0.58	Tail torsion

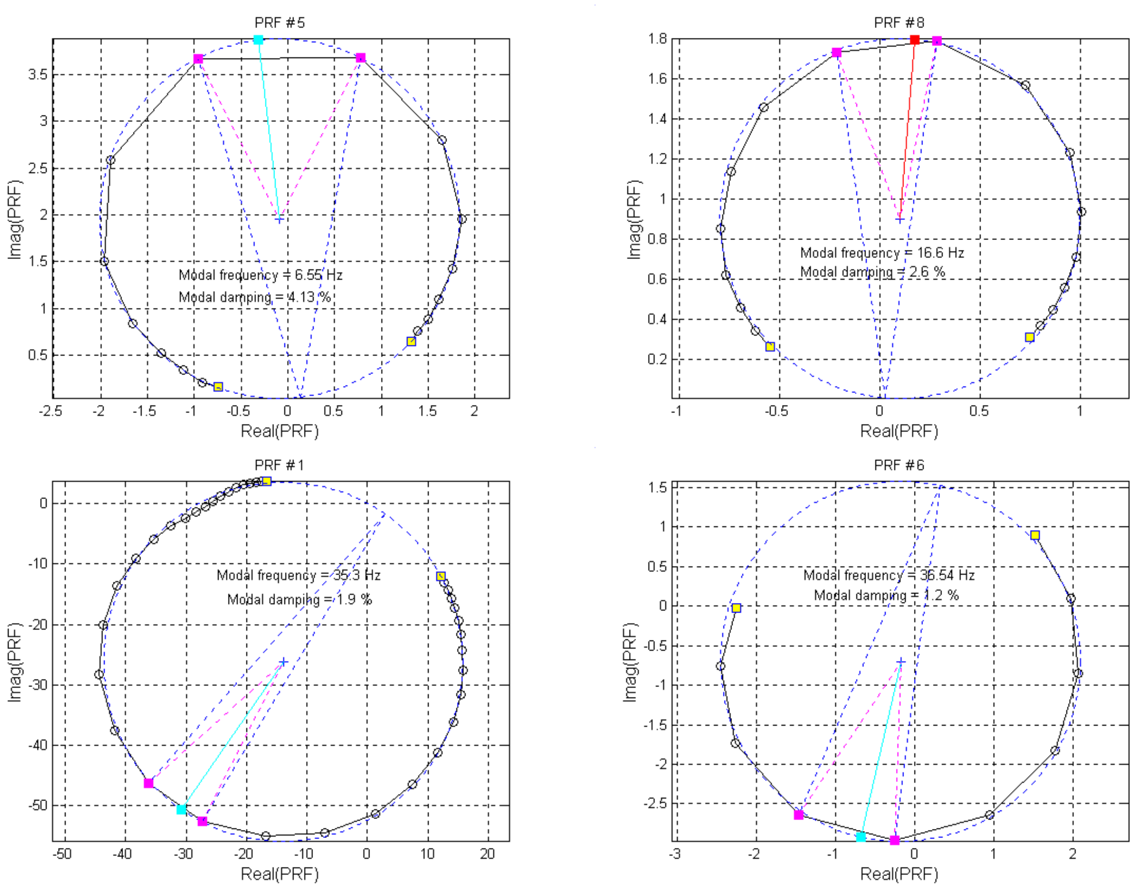


Fig. 5: SDOF circle fit to PRFs

Examples of circle fit modal analysis are illustrated in Fig.5. The almost circular shape of PRFs in the neighbourhood of resonance indicates a good mode isolation for SDOF analysis.

Modal viscous damping values are calculated as the arithmetic mean of two values, one determined using the two points chosen next to resonance, indicated in figures, the other determined using the next close points below and above resonance.

Figure 6 presents the MIF and ImMIF plots. Both fail to indicate all three close modes at about 35 Hz.

Figure 7 presents the PRFMIF and the complementary 1-PRFMIF computed for a rank N_r =11. In the figure they are denoted AMIF and 1-AMIF, respectively, as in [4]. They locate only two of the three modes at 35 Hz.

All four MIFs from Fig.6 and Fig.7 have limits, being single-curve indicators. They can be used in a first stage of the analysis to locate not too close modes and are best suited for structures with many modes within the frequency band of interest. A check for close modes should be carried out in a second stage. The CoMIF plot helps locating all active modes. Though the number of PRFs exceeds the number of active modes, there are no computational modes to be sorted out.

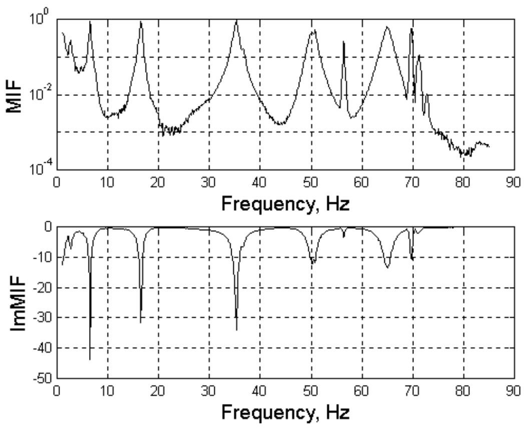


Fig. 6: MIF and ImMIF

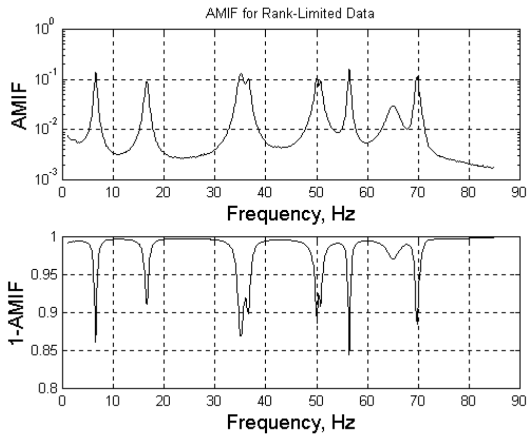


Fig.7: PRFMIF and 1-PRFMIF

CONCLUDING REMARKS

Based on the new measurements on the GARTEUR SM-AG19 Testbed, the paper shows how the Principal Response Analysis can be used for modal parameter identification from a restricted and non-optimal data set. It is a pleading for the use of two new mode indicator functions, the CoMIF and the PRFMIF, to determine the number of modes present in a given frequency range. Based on the information density of the data set, they have a different physical background and outperform the commonly used MIFs, developed to locate frequencies where the response is close to the monophase condition.

Basically, PRA separates the system response into incoherent components. Its aim is to replace the measured set of FRFs by a reduced set of uncorrelated PRFs. A rank-limited set of virtual FRFs can be reconstructed from the truncated set of PRFs. Their analysis is not recommended, because they represent a structure constrained by the cancellation of several singular values. Instead, analysis of isolated peaks in individual PRFs yields accurate results using simple SDOF modal parameter identification procedures.

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