USE OF RECIPROCAL MODAL VECTORS FOR NON-LINEARITY DETECTION

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Use of Reciprocal Modal Vectors for Non-linearity Detection

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Summary

This paper seeks to exploit the reciprocal modal vector orthogonality between experimentally-derived mode shapes and the corresponding measured frequency response functions in order to derive a criterion for the detection of structural non-linearities. Being based on the use of measured data and experimentally-derived quantities only, the method is directly applicable to practical engineering cases for which there are usually no spatial descriptions. A brief outline of the reciprocal modal vector theory is given first, followed by a short description of a frequency-domain non-linear response simulation technique using a harmonic balance approach. A detailed study of a 4-DOF system with cubic stiffness and friction damping non-linearities is presented next. It is shown that the proposed non-linearity detection criterion is relatively insensitive to measurement noise. It is concluded that the method is effective for detecting non-linear behaviour in a consistent and quantitative fashion.

Keywords: Experimental modal analysis, non-linear frequency response, harmonic balance, friction damping, cubic stiffness

1. Introduction

Non-linear behaviour in structural dynamics is somewhat difficult to define in a general manner. From an engineering standpoint, a system is considered to be non-linear if its dynamic properties such as stiffness and/or damping depend on displacement, velocity, acceleration, or any combination of these variables. The general problem of studying the dynamics of a non-linear system is further compounded by the fact that non-linear behaviour cannot be isolated from the operating range and conditions which give rise to it in the first place. Considering that such conditions may encompass a wide range of situations, a complete analysis may become very expensive, if not impracticable.

The detection, identification and quantification of structural non-linearities have been the subject of many research papers. Several review articles reveal a myriad of techniques and seem to indicate that the choice of a particular algorithm is closely linked to the objectives of the analysis **[1, 2, 3, 4 & 5]**. Although it is generally accepted that, for a certain class of data, it is possible to follow a certain procedure to obtain acceptable results, it is also recognised that such findings cannot be generalised. Such considerations highlight the need for a generally-applicable method that is also compatible with the current practice in experimental modal analysis. In most experimental studies, the measured frequency response functions (FRFs) are routinely curve-fitted to extract the structure's modal parameters by assuming that its dynamic behaviour is linear. However, all practical structures are, to a certain extent, non-linear. Therefore, from a practitioner's viewpoint, it is highly desirable to be able to correlate the results of standard linear modal analysis and raw response data in order to assess the extent and nature of the existing non-linearities. The paper will describe an orthogonality-based methodology towards this aim.

Orthogonality is a familiar concept in structural dynamics. Some orthogonality properties of MDOF systems, such as eigenvector orthogonality with respect to the system's mass and stiffness matrices, are well known while other forms are less common but have the potential of being equally useful. This paper will explore a particular form of orthogonality, that between experimentally-derived mode shapes and the so-called reciprocal modal vectors (RMVs), which inherently exists in all linear MDOF systems. The concept of RMV is first introduced in **[6]** where it was defined as an orthogonality criterion for experimental modal analysis. The work was extended by in **[7]** where the orthogonality between an identified mode shape and measured FRF data was checked. The RMV orthogonality was also used as a criterion to assess the quality of the modal analysis results **[8]**. It was demonstrated that such a quality assessment of modal analysis results could be performed using measured FRF data only. The main advantage of the method is the fact that it does not require the (usually unavailable) mass matrix of the system so that it can be used in practical engineering cases. The purpose of the present work is to extend the use of RMV criterion to non-linear systems so that it can be applied to the detection of structural non-linearities.

2. BASIC THEORY

2. 1 Reciprocal modal vector (RMV) formulation

The matrix equation of motion for a linear N degree-of-freedom system can be written as:

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}+=\{0\} \qquad (1)$$

where [M], [K], [C] are the symmetric mass, stiffness and damping matrices of the system and {x} is the vector of displacements. Under the assumption of simple harmonic motion, the system's natural frequencies ω_r and mode shapes $\{\phi\}_r$ can be found via a standard eigensolution, where r is the vibration mode index.

The reciprocal modal vector (RMV) was defined as the product of the mass matrix and the mode shape vector **[6]**:

$$\{\chi\}_r = [M]\{\phi\}_r \qquad (r=1,2,....,N) \qquad (2)$$

By virtue of modal orthogonality with respect to the mass matrix, one can write:

$$\{\phi\}_i^T\ [M]\{\phi\}_j = \{\chi\}_i^T\ \{\phi\}_j = \{\phi\}_i^T\ \{\chi\}_j = \delta_{ij} \qquad (3)$$

where δ_{ij} is the Kronecker's delta.

In matrix form, (2) can be written as:

$$[\chi]^T[\phi] = [\phi]^T[\chi] = [^{\cdot}I_{\cdot}] \ . \qquad (4)$$

where $[^{\cdot}I_{\cdot}]$ is the unit matrix. It can now be seen that the physical basis of the modal reciprocal vector is the orthogonality of the normal modes for linear systems. Indeed, (4) is a variant of the classic orthogonality equation $[\Phi]^T[M][\Phi] = [^{\cdot}I_{\cdot}]$,

In practice, the experimental RMV cannot be calculated from (2) since the mass matrix is not usually available. However, as formulated in **[7]**, use can be made of the measured FRF data to obtain the RMV. The *p*-th column of the experimental receptance matrix can be expressed as:

$$\{\alpha(\omega)\}_p = \left\{ \sum_{r=1}^{N} \frac{\phi_{pr}\,\phi_{1r}}{\omega_r^2 - \omega^2}\ , \sum_{r=1}^{N} \frac{\phi_{pr}\,\phi_{2r}}{\omega_r^2 - \omega^2}\ , \ldots, \sum_{r=1}^{N} \frac{\phi_{pr}\,\phi_{Nr}}{\omega_r^2 - \omega^2} \right\}^T$$

$$= \frac{\phi_{p1}}{\omega_1^2 - \omega^2} \{\phi\}_1 + \frac{\phi_{p2}}{\omega_2^2 - \omega^2} \{\phi\}_2 + ... + \frac{\phi_{pN}}{\omega_N^2 - \omega^2} \{\phi\}_N \quad (5)$$

In a practical situation, there will be N measured FRFs, leading to an extracted mode shape vector of length N for a given vibration mode. The selection of the N measurement points is not straightforward and it is well outside the scope of this paper. However, once such a choice is made, the orthogonality of the eigenvector with N elements is assessed against N measured FRFs.

In any case, it can easily be seen that (6) is written in terms of directly-measured (raw FRFs) or experimentally-derived quantities (linear modal parameters) only. Re-arranging (5) gives:

$$\{\alpha(\omega)\}_p^T \{\chi\}_r = \frac{\phi_{pr}}{\omega_r^2 - \omega^2} \quad . \quad (6)$$

For a given frequency range, the RMV can be determined by writing equation (6) at *s* excitation frequencies and hence forming a set of linear equations for the unknown elements vector $\{\chi\}_r$. Such an approach yields:

$$[\{\alpha(\omega_1)\}_p, \{\alpha(\omega_2)\}_p, ... \{\alpha(\omega_s)\}_p]^T \{\chi\}_r = \phi_{pr} \begin{Bmatrix} \frac{1}{\omega_r^2 - \omega_1^2} \\ \frac{1}{\omega_r^2 - \omega_2^2} \\ ... \\ \frac{1}{\omega_r^2 - \omega_s^2} \end{Bmatrix} \quad (7)$$

A close inspection of (7) reveals the fact that the FRF matrix on the left-hand side may have more frequency points than actual measurement points in most cases. In other words, (7) can be made over-determined by a suitable selection of the number of excitation points *s*. In the numerical studies of **Section 3**, the singular value decomposition (SVD) technique was used to obtain the RMV matrix from (7). This approach has the added advantage of removing the unwanted noise in the data and finding the best fitting to the given data. Once an RMV is determined, it can be used to assess the orthogonality between the mode shape extracted via modal analysis and the original FRF data subjected to modal analysis. For a linear system and noise-free measurements, all elements of the RMV should be zero, except the element corresponding to the mode shape under study, which should be unity. Deviations of the RMV matrix from this pattern may be caused by noise in the measurements, systematic errors due to experimental set-up, poor modal analysis or structural non-linearities. In the next section, an attempt will be made to distinguish the effects of these different sources.

2.2 Generation of reference non-linear response data

There are several methods to simulate the response of non-linear systems. Here our purpose is to generate some reference non-linear response data, perform a linear modal analysis and see if it is possible to detect the non-linear behaviour by inspecting the reference data and the results of the linear modal analysis via the RMV method. For convenience, the non-linear response of a MDOF system will be simulated in the frequency domain using an equivalent linearisation technique where the system is assumed to behave linearly for a given set of conditions. These conditions are changed iteratively to determine the non-linear behaviour as a series of linear cases. The most commonly used variant of the equivalent linearisation method, the first-order harmonic balance, is a fast and accurate technique, provided that the higher harmonic terms remain small relative to the fundamental component. The technique is based on calculating the first harmonic component of the non-linear force in terms of an equivalent linearised stiffness and/or damping, though multi-harmonic versions also exist **[9]**. The focus of the work is to study the manner in which a non-linear system will violate the orthogonality between extracted modal parameters and measured FRFs when a linear modal analysis technique is used. Therefore, the simulation of the non-linear response by using the first harmonic only is a matter of convenience. If a non-linear response with more harmonics is used, the distortion in the RMVs is likely to be larger, thus making the detection of non-linearity easier.

The harmonic balance method is essentially an optimisation process which seeks an equivalent linear model to a non-linear second order dynamic system in a such way that the average of the difference between the two is minimised. The method only works for relatively weak non-linearities and relies on the assumption of harmonic response. The equivalent linear stiffness can be determined by calculating the first few harmonic terms of the non-linear force over one vibration cycle **[10], [11]**.

In the case of a non-linear system, subjected to a harmonic excitation with frequency Ω, (1) becomes:

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}+\{f_{nl}(x,\dot{x})\}=\{F\}\cos\Omega t \tag{8}$$

where the vector $\{f_{nl}(x,\dot{x})\}$ represents the non-linear terms which are a function of displacement and velocity. Non-linearities arising from inertia effects are omitted from (8).

Assuming harmonic oscillation and considering the first harmonic only, the real and imaginary parts of a linearised equivalent stiffness can be expressed as:

$$K_r^{'}=\frac{1}{\pi x}\int_0^{2\pi} f_{nl}\cos\Omega t\,dt \qquad K_i^{'}=-\frac{1}{\pi x}\int_0^{2\pi} f_{nl}\sin\Omega t\,dt \tag{9}$$

Each element of the $\{f_{nl}(x,\dot{x})\}$ vector can be processed using (9) and the result can be substituted into (8) in the form of a linearised stiffness matrix [K’]. The non-linear response can then be computed iteratively from:

$$\{x\}=[\alpha(x)]\{F\} \tag{10}$$

where the receptance FRF matrix, [α], can explicitly be written as:

$$[\alpha(x)] = ([K]+[K'(x)]+i\Omega[C]-\Omega^2[M])^{-1} \tag{11}$$

3. CASE STUDY

The 4 DOF system of **Fig. 1** was used throughout the case studies and the first column of the FRF matrix was assumed to have been measured for a frequency range covering all 4 modes. The modal properties of the system are listed in **Table 1**.

3.1 Linear noise-free FRFs

A column of linear, noise-free FRFs was generated using the modal parameters of **Table 1** and a structural damping value of 0.5%. During the simulation, mass 1 was excited and the response was obtained at all four masses. The modal parameters, i. e. the natural frequencies, the damping factors and the mode shapes were extracted using a standard modal analysis package. The matrix of the RMVs was then computed using (7). The RMVs, listed in **Table 2**, yield the expected pattern of a unit matrix.

3.2 Linear FRFs polluted by simulated experimental noise

The measurement noise was simulated using a simple model given by:

$$\alpha' = \alpha(1 + n\varepsilon) \tag{12}$$

where α' is the noise-polluted response, ε is a uniformly-distributed random number between 0 and 1, and n is a noise percentage. As before, the polluted FRFs were subjected to modal analysis and the RMVs were obtained using (7). Although some very small irregularities were observed, the RMV matrix remained, in the main, a unit matrix for up to about 20% added noise. This result can be explained by the noise averaging effects of the two main numerical procedures involved: the modal analysis for obtaining the modal properties and the SVD inversion that is used for solving (7). As can be seen from **Table 3**, the overall shape of the RMV matrix remains that of a unit matrix for 20% added noise, maximum deviations being under 2.5%. Remembering that experimental noise is unlikely to exceed 10%, the RMV matrix can be considered to be insensitive to measurement noise.

3.3 Modal analysis errors

In real engineering applications, it is likely that the modal properties will be polluted by analysis errors due to relatively poor curve-fits, inconsistencies in measured data, etc. Such a situation was simulated by randomly polluting the modeshape vectors by up to 10% and computing the RMV matrix for the polluted modal properties. Once again, the RMV matrix was found to be relatively insensitive to modal analysis errors, the largest off-diagonal elements remaining below 4%. The deviation from the unit matrix, defined as the RMV matrix minus the unit matrix, is plotted in **Fig. 2**. For visualisation purposes, each RMV is plotted in a different shade of grey.

3.4 Application of the RMV technique to non-linear FRFs

For a non-linear structure, the modal parameters will depend on the response amplitude. As such, a linear modal analysis, which assumes a constant mode shape for a mode, will inevitably yield somewhat inaccurate results. However, provided the extent of the non-linearity is small or the response levels are kept low, linear modal analysis can still be used for the extraction of modal parameters with reasonable accuracy. The aim of here is to focus on the disturbance of the RMV matrix due to non-linear effects.

3.4.1 Cubic stiffness non-linearity

Using the same 4 DOF system again, a cubic stiffness type non-linearity was introduced between masses 2 and 3. The force-displacement characteristic is given by:

$$F = \beta x^3 \tag{13}$$

An inspection of the modeshapes listed in **Table 1** reveals that only modes 2 and 4 will be affected as masses 2 and 3 move in opposite direction for both modes. As a first numerical test, responses with relatively weak non-linear effects were obtained for a 50N force using the harmonic balance method of **Section 2.2**. As before, these non-linear responses were subjected to a linear modal analysis for parameter extraction. Using the extracted modal parameters, it is possible to regenerate the response and overlay it against the original non-linear FRF (**Fig. 3**). It is immediately seen the original non-linear response and the regenerated (linear) response are not identical, the discrepancy being due to non-linear effects.

In order to assess whether the non-linear effects can be detected by the proposed orthogonality-based method, the RMV matrix was computed using (7). The results, given in **Table 4**, indicate that only the 2nd and the 4th RMVs are affected, the other two remaining unaffected. The main diagonal entries corresponding to modes 2 and 4 are 84% and 90% respectively, these values representing much larger changes than those produced by noise or modal analysis errors.

The deviation from the unit matrix is plotted in **Fig. 4**. Unlike the previous cases, the changes have a distinct pattern and there are no random variations. The fact that the RMV matrix is able to detect the non-linearity is an encouraging result as this feature is not evident from the response plots of **Fig. 3**.

The force level was increased to 150N and a new set of simulated non-linear responses was obtained. The deviation of the corresponding RMV matrix from the unit matrix is plotted in **Fig 5**. As before, only the 2nd and 4th modes are affected, but now to a much greater extent. This is important finding, relating the change in the RMV matrix to the amplitude of input force. Therefore, it is possible to focus on non-linearity effects by considering different levels of input force and by observing the variation of the RMV matrix. Although it can be argued that a similar visual inspection can be performed by considering the raw FRF data acquired at different force levels, the inspection of distortions around resonances is somewhat more subjective and less quantitative.

3.4.2 Friction damping non-linearity

A similar case study was also conducted with friction damping non-linearity. A micro-slip model, which considers the effect of partial slip at some parts of the metal-to-metal interface before the gross slip occurs, was used in the formulation. The relationship between the load (F) and the displacement (x) can be expressed as **[12]**:

$$F = \begin{cases} s\mathrm{x} - t\mathrm{x}^2 & if \quad 0 < \mathrm{x} < \dfrac{s}{2t} \\ \dfrac{s^2}{4t} & \quad \mathrm{x} \geq -\dfrac{s}{2t} \end{cases} \tag{14}$$

where parameters s and t depend on several factors, such as the effective contact area, surface asperity, friction coefficient, etc. As shown in **Fig. 6**, the load curve of the micro-slip model under a cyclic load exhibits a hysteresis loop.

In this case, the friction damping non-linearity was introduced between mass 1 and ground. The original cubic stiffness non-linearity was removed. As before, the first column of the non-linear FRF matrix was computed for a given excitation level and appropriate friction damper model parameters. A linear modal analysis was performed for parameter extraction. The deviation of the RMV matrix from the unit matrix is shown in **Fig. 7**.

The findings are consistent with the previous results in the sense that a friction damper between mass 1 and ground is likely to affect the first two modes. The RMVs of the remaining modes are unchanged.

3.4.3 Combined cubic stiffness and friction damping non-linearity

As a final numerical test, the cubic stiffness element of **Section 3.4.1** was re-introduced into the 4-DOF model. The deviation from the unit matrix is plotted in **Fig. 8** for the resulting system with two types of non-linearity. As expected, the magnitude of element (2,2) of the deviation matrix has increased substantially due to combined non-linear effects in that particular mode. Similarly, the non-linear behaviour detected in Mode 4 is due to the addition of the cubic stiffness element and the corresponding RMV can be traced back to **Fig. 5**.

4. CONCLUDING REMARKS

(i) Experimentally-derived mode shapes and their reciprocal modal vectors are shown to be orthogonal to each other. Such a feature allows the derivation of a detection matrix which becomes coincident with a unit matrix for linear, noise-free systems.

(ii) The RMV method can be exploited to detect non-linear behaviour from a linear modal analysis of the measured responses. A direct application can be found in experimental modal analysis where it is common practice to subject measured FRFs to linear modal analysis for system identification purposes. An inspection of the RMVs offers a practical and objective means of assessing the results of modal analysis, not only for non-linearity detection but also to spot any other analysis errors. The orthogonality properties will have been violated in both cases.

(iii) For linear systems, the matrix of reciprocal modal vectors is found to be relatively insensitive to noise effects or small modal parameter errors. Such features make it suitable for the detection of non-linear behaviour when other errors can be considered to be small.

(iv) For non-linear systems, the RMV matrix is found to be a good indicator of modal non-linearity. The numerical implementation is straightforward and requires little computational effort. The method is particularly suited to practical engineering cases because of being compatible with existing linear modal analysis techniques and of using measured data only. The deviation from the unit matrix has a distinct pattern: leading diagonal elements corresponding to the mode(s) exhibiting non-linear behaviour and the cross-terms. Furthermore, the changes are approximately proportional to the level of force applied.

(v) The technique is likely to work better with separated modes, a feature which will ensure that the modal analysis errors will be small. In some difficult cases involving close modes, the RMVs may well be contaminated with modal analysis errors.

(vi) The proposed detection technique appears to work when there are combined non-linearities, here cubic stiffness and friction damping. However, the results are based on a small-size system with good numerical properties. Further studies with larger systems and experimental verification are considered to be necessary.

(vii) Further studies, not reported here, have shown that the magnitude of the RMVs is related to the amount of non-linearity present in the system. Therefore, the proposed detection technique can also be used as a quantification tool.

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Table 1 - Modal model for the system of **Figure 1**

	Mode 1	Mode 2	Mode 3	Mode 4
Natural freq. (Hz)	8.61	14.0	20.8	28.6
Co-ordinate 1	0.3536	-0.4756	0.3536	0.1515
Co-ordinate 2	0.5000	-0.2142	-0.5000	-0.6739
Co-ordinate 3	0.5000	0.2142	-0.5000	0.6739
Co-ordinate 4	0.3536	0.4765	0.3536	-0.1515

Table 2 – RMVs for linear noise-free FRFs

Mode 1	Mode 2	Mode 3	Mode 4
0.9960	0.0001	0.0000	0.0000
0.0000	0.9978	0.0000	0.0000
0.0000	0.0000	0.9914	0.0000
0.0001	0.0000	0.0000	0.9982

Table 3 – RMVs for linear FRFs with 20% noise

Mode 1	Mode 2	Mode 3	Mode 4
0.9932	0.0159	0.0237	0.0064
0.0034	0.9771	0.0049	0.0066
0.0086	0.0173	0.9889	0.0090
0.0030	0.0190	0.0097	1.0042

Table 4 – RMVs for weak non-linearity (F=50N)

Mode 1	Mode 2	Mode 3	Mode 4
0.9992	0.0006	0.0000	0.0000
0.0000	**0.8409**	0.0000	0.0018
0.0000	0.0019	0.9915	0.0001
0.0000	0.0260	0.0001	**0.9008**

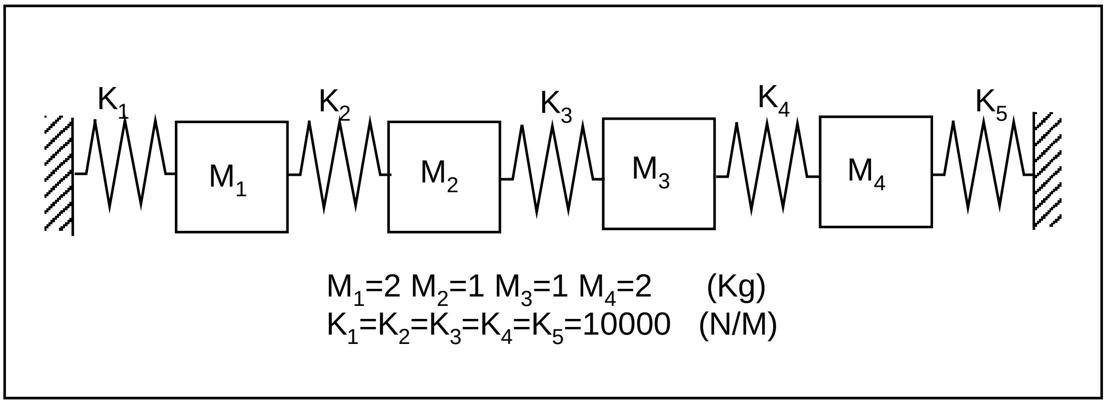

Fig. 1. The 4 DOF system

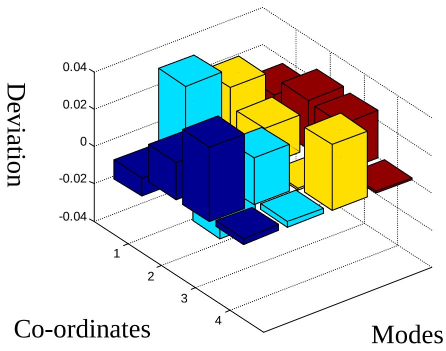

Fig. 2 Deviation of the RMV matrix from the unit matrix – Modal analysis errors

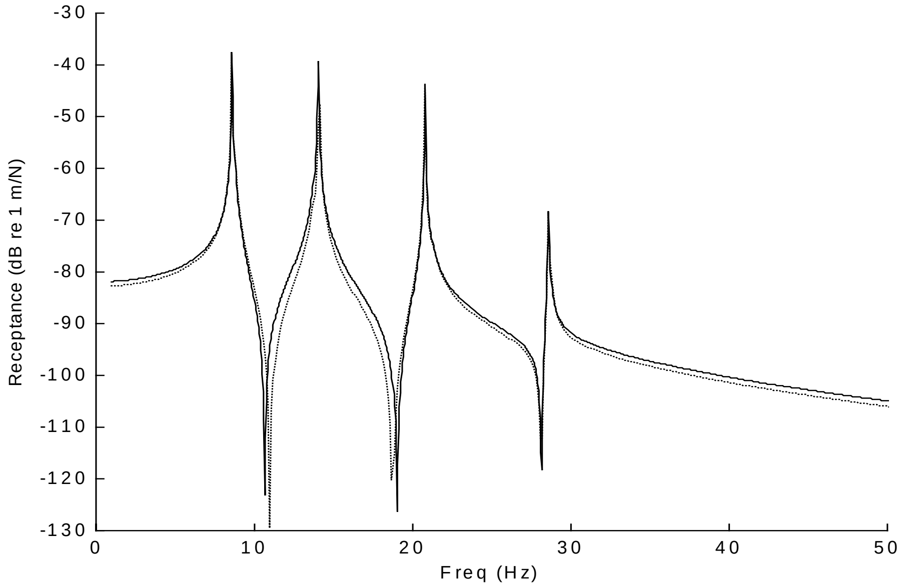

Fig. 3 Non-linear response (solid line) vs. regeneration (dotted line) using parameters from linear modal analysis (F=50N)

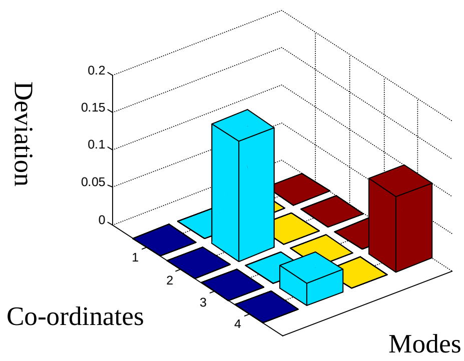

Fig. 4 Deviation of the RMV matrix from the unit matrix – Low force level

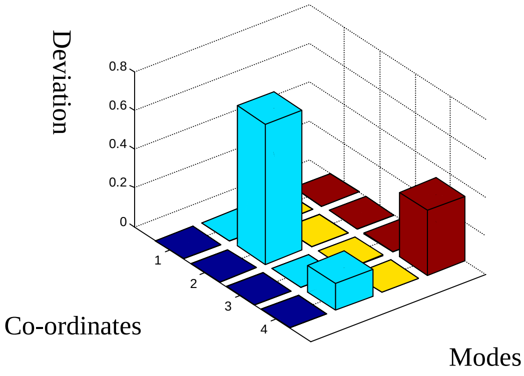

Fig. 5 Deviation of the RMV matrix from the unit matrix – High force level

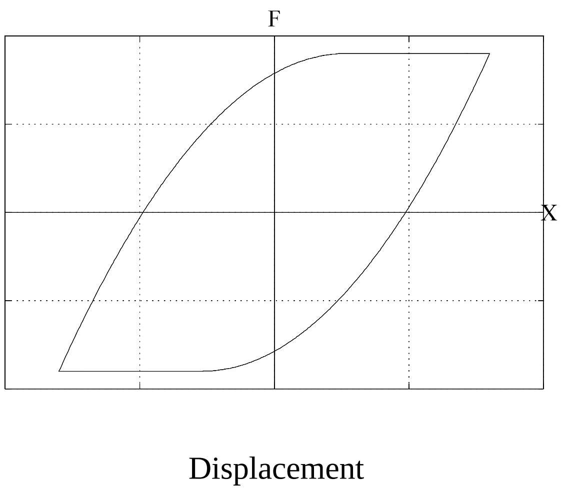

Fig. 6 Micro-slip model (Burdekin et al. 1978)

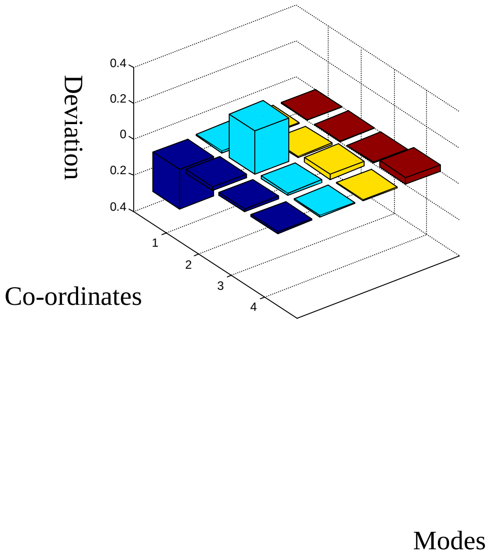

Fig. 7 Deviation of the RMV matrix from the unit matrix – Friction damping non-linearity

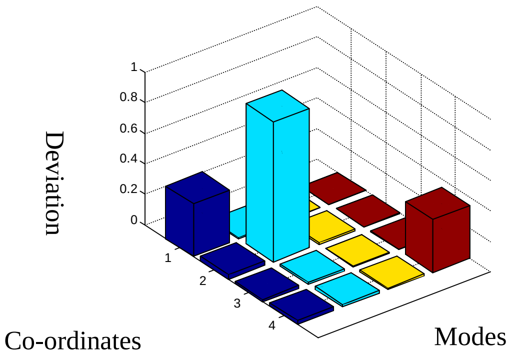

Fig. 8 Deviation of the RMV matrix from the unit matrix – Combined non-linearity